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Uniqueness of Nash Equilibrium in Private Provision of Public Goods: Extension

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Abstract

This note proves uniqueness of Nash equilibrium in private provision of public goods under the general economy with endogenous labor supply, distortionary income tax with taxable limit and distortionary national subsidies.

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1. Introduction

Many papers have analyzed the effect of income redistribution[1], national subsidies[2]and voluntary subsidies[3] under the economy with private provision of public goods. However most of them have not considered the uniqueness problem of Nash equilibrium achieved in the economy. For example, Bernheim (1986, 1992) proposed the neutrality of income redistribution under the general economy with endogenous labor supply and distortionary income tax. However the general neutrality holds only if there exists only one Nash

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[1] Warr (1983), Roberts (1984), Bernheim (1986), Bergstrom et. al. (1986), Bernheim and Bagwell (1988), Andreoni (1988, 1989, 1990), Boadway et. al. (1989a), Sandler and Posnett (1991), Fries et. al. (1991), Glazer and Konrad (1993), Buchholz and Konrad (1995) and Ihori (1996) focus on the effect of the income transfer among agents. On the other hand, Warr (1982), Bergstrom et. al. (1986), Steinberg (1986, 1987), Roberts (1987), Bernheim and Bagwell (1988), Andreoni (1988, 1990) and McClelland (1989) focus on the income transfer between agents and the government..

[2] For example, see Feldstein (1980, 1987), Driessen (1987), Bergstrom (1989), Boadway, Pestieau and Wildasin (1989a, 1989b), Roberts (1992), Andreoni and Bergstrom (1996), Ihori and Akai (1996), Cremer et. al. (1997) and Kirchsteiger and Puppe (1997)..

[3] For example, see Buchholz and Konrad (1995)

equilibrium in voluntary provision of public goods. Bergstrom, Blume and Varian (1986) and Andreoni and Bergstrom (1996) have proved uniqueness of equilibrium. However they focused on the economy with the fixed labor supply.

In order to derive results from possible analysis exactly, it is important to prove uniqueness of Nash equilibrium in private provision of public goods under the general economy with endogenous labor supply, distortionary subsidy and distortionary income tax with taxable limit. [4]

2. Model

In this paper, we consider an economy with *I* agents, indexed by *i*. We assume that agents get utility from the private good consumption, leisure and the public good continuously. Agents are assumed to be pure altruistic with respect to their contribution, in the sense that they are interested in the total amount of the provision of public goods. Then the utility function can be defined as the following function;

$$U^i = u^i(X^i, -l^i, D^i + \sum_{j \neq i} D^j),$$

where X^i and $-l^i$ represent the levels of consumption of private goods and consumption of leisure, respectively. l^i means the level of labor supply when the available time is fixed. In addition, D^i and $\sum_{j \neq i} D^j$ represents the own contribution of public goods and the total amount of others' contribution, respectively.

Next let us consider the budget constraint of agents. We assume that there exists the linear income tax with a taxable limit in the economy. The income tax rate is defined as t and the taxable limit is z. Assuming that the wage rate is w, the income of agent *i* becomes wl^i. When the government subsidizes the contribution to the public good by agents, the budget constraint of agent *i* is described as[5]

$$X^i + q(1-\mathrm{b})D^i = wl^i - t(wl^i - z) \tag{1}$$

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[4] It is also important to prove the existence of Nash equilibrium. However the proof of the existence is a straightforward application of Brouwer's fixed point theorem. See, for example, Bergstrom, Blume and Varian (1986) and Andreoni and Bergstrom (1996).

[5] This setting includes the economy which has the deduction of the contribution from the income tax base. When we define the deduction rate as a, the budget constraint can be written as . $X^i + qD^i = w^i l^i - t(w^i l^i - z - \mathrm{a}qD^i)$. When we set $\mathrm{b} = t\mathrm{a}$, this constraint becomes the same as equation (1).

where b represents the subsidy rate such that $0 \le b \le 1$ and q represents the price of contribution. This equation can be rewritten as

$$X^i + pD^i = (1-t)wl^i + tz$$

where p is defined as the net price of the contribution of public goods which the agent faces, $q(1-b)$.

Finally the budget constraint of the government is as follows;

$$\sum qbD^i = \sum t(wl^i - z)$$

3 Uniqueness of Nash equilibrium

The following main theorem holds in the general economy.

Theorem

If preferences are strictly convex and both the public and private goods are normal goods, then there exists exactly one Nash Equilibrium of private provision of public goods, $D^1, \ldots\ldots D^I$.

Proof

The budget constraint of each agent is represented as follows.

$$X^i + pD^i = (1-t)wl^i + tz$$

Considering others' contribution into both side of above equation, we have

$$X^i + p\sum D^i = (1-t)wl^i + tz + p\sum_{j \ne i} D^j ,$$

which becomes

$$X^i + p\sum D^i + (1-t)w(-l^i) = m^i ,$$

where $m^i = tz + p\sum_{j \ne i} D^j$.

Let demand functions $\mathbf{D}^i$ $(p,(1-t)w,m^i)$, $X^i(p,(1-t)w,m^i)$ and $l^i(p,(1-t)w,m^i)$ be $\sum D^i$, X^i and l^i respectively that maximize $u^i(X^i, -l^i, \sum D^i)$ subject to $X^i + p\sum D^i + (1-t)w(-l^i) = m^i$. We assume that X^i, $\mathbf{D}^i$, and $-l^i$ are normal goods, that is, these three demand functions are strictly increasing with respect to m^i. First, the following two lemmas are useful in proving uniqueness.

Lemma A

Suppose that for a given subsidy rate, tax parameters, there are two distinct Nash equilibria $D^1, \ldots\ldots D^I$ and $D'^1, \ldots\ldots D'^I$ ($I \ge 2$), such that $\sum D'^i \ge \sum D^i$ and for some i $D^i \neq D'^i$. Then, for all i, $\sum_{j \neq i} D'^j \geq \sum_{j \neq i} D^j$.

Proof of Lemma A

Consider two cases where $D'^i > 0$ and $D'^i = 0$.

(Case 1) Consider the case where $D'^i > 0$. Then since $D'^1, \ldots\ldots, D'^I$ is Nash equilibrium, it must be that $\sum D'^i = \mathbf{D}^i(p,(1-t)w, tz + p\sum_{j \neq i} D'^j)$. Since $D^1, \ldots\ldots, D^I$ is Nash equilibrium, it must be that $\sum D^i \geq \mathbf{D}^i(p,(1-t)w, tz + p\sum_{j \neq i} D^j)$.[6] By assumption, $\sum D'^i \geq \sum D^i$. Therefore $\mathbf{D}^i(p,(1-t)w, tz + p\sum_{j \neq i} D'^j) \geq \mathbf{D}^i(p,(1-t)w, tz + p\sum_{j \neq i} D^j)$. Since $\mathbf{D}^i$ is assumed to be a normal good, we get $\sum_{j \neq i} D'^j \geq \sum_{j \neq i} D^j$.

(Case 2) Consider the case where $D'^i = 0$. Then we get that $\sum_{j \neq i} D'^j = \sum D'^i$. Since $\sum_{j \neq i} D^j \leq \sum D^i$ and $\sum D'^i \geq \sum D^i$, we get that $\sum_{j \neq i} D'^j \geq \sum_{j \neq i} D^j$. **Q. E. D.**

Lemma B

For two Nash equilibria assumed in Lemma A, it holds that, for some i, $\sum_{j \neq i} D'^j > \sum_{j \neq i} D^j$.

Proof of Lemma B

Suppose that for all i, $\sum_{j \neq i} D'^j = \sum_{j \neq i} D^j$. Summing it up over consumers, we have $\sum_i^I \sum_{j \neq i} D'^j = \sum_i^I \sum_{j \neq i} D^j$, that is, $(I-1)\sum_i^I D'^i = (I-1)\sum_i^I D^i$. Since I is more than one, we get that $\sum D'^i = \sum D^i$ for all i.

On the other hand, by definition, $\sum_{j \neq i} D'^j = \sum D'^i - D'^i$ and $\sum_{j \neq i} D^j = \sum D^i - D^i$ for all i. Since $\sum D'^i = \sum D^i$ for all i, it must be that $D'^i = D^i$ for all i. This contradicts

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[6] If $D^i > 0$, $\sum D^i$ becomes equal to $\mathrm{D}^i(p,(1-t)w, tz + p\sum_{j \neq i} D^j)$ since the agent supplies the public good voluntarily such that $\sum D^i$ becomes equal to the desirable amount for the agent , that is, $\mathrm{D}^i(p,(1-t)w, tz + p\sum_{j \neq i} D^j)$. On the other hand, if $D^i = 0$, $\sum D^i$ is not necessarily equal to the desirable one because D^i can not be negative.

assumption in Lemma A that two Nash equilibria are distinct. Therefore it must be that, for some *i*, $\sum_{j\neq i} D'^j > \sum_{j\neq i} D^j$. **Q. E. D.**

Next, in order to prove uniqueness, suppose that two distinct Nash equilibria corresponding to a given parameter. Let $D^1,\dots\dots D^I$, $X^1,\dots\dots X^I$, $l^i,\dots\dots l^I$ and $D'^1,\dots\dots D'^I$, $X'^1,\dots\dots X'^I$, $l'^i,\dots\dots l'^I$ be the individual contribution, the private consumption and labor supply in the two equilibria. Suppose, without loss of generality, that $\sum D'^i \geq \sum D^i$. Then according to Lemma A, it must be that for all *i*, $\sum_{j\neq i} D'^j \geq \sum_{j\neq i} D^j$. Since the private good and leisure are normal goods, Lemma B gives that $X'^i = X^i\,(p,(1-t)w,tz+p\sum_{j\neq i} D'^j) \geq X^i\,(p,(1-t)w,tz+p\sum_{j\neq i} D^j) = X^i$ and $l'^i = l^i\,(p,(1-t)w,tz+p\sum_{j\neq i} D'^j) \leq l^i\,(p,(1-t)w,tz+p\sum_{j\neq i} D^j) = l^i$ for all *i* with strictly inequality for some *i*. Therefore it must be that $\sum X'^i > \sum X^i$ and $\sum l'^i < \sum l^i$. Since the feasibility condition means that $\sum X^i + q\sum D^i - w\sum l^i = 0$, we get that $\sum X'^i + q\sum D'^i - w\sum l'^i = 0 = \sum X^i + q\sum D^i - w\sum l^i$. Since $\sum X'^i > \sum X^i$ and $\sum l'^i < \sum l^i$, it must be that $\sum D'^i < \sum D^i$, which contradicts assumption that $\sum D'^i \geq \sum D^i$. Therefore there never exist two distinct Nash equilibria, that is, there exists unique Nash equilibrium. **Q. E. D.**

4. Conclusion

This paper has proved uniqueness of Nash equilibrium in private provision of public good in the general setting which allows the existence of any distortionary income tax with taxable limit, any subsidy and endogenous labor supply. This proof has extended the results of Bergstrom, Blume and Varian (1986) and Andreoni and Bergstrom (1996). In order to discuss the effect of the public policy in private provision of public goods exactly, the proof of uniqueness of Nash equilibrium in the general setting is important and useful for the future analysis.

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