

## Optimality of a Competitive Equilibrium in a Small Open City with Congestion\*†

Nobuo Akai

*The Institute of Economic Research, Kobe University of Commerce  
8-2-1 Gakuen-nishimachi Nishi-ku, Kobe 651-21 Japan*

Takashi Fukushima‡

*Faculty of Economics, Tokyo Metropolitan University,  
1-1 Minami Osawa, Hachioji, Tokyo 192-03, Japan*

and

Tatsuo Hatta

*Institute of Social and Economic Research, Osaka University,  
6-1 Mihogaoka, Ibaraki, Osaka 567, Japan*

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Consider a small spatial economy with a congestion externality in the public transportation sector. We prove that the competitive equilibrium allocation coincides with the social optimum allocation despite the presence of the congestion externality. To this end, we first specify a set up conditions for a competitive equilibrium allocation, and then we find another set of conditions for social optimality. By comparing the two set of conditions, we prove that the competitive allocation is compatible with the social optimal allocation. We would like to emphasize the fact that no optimal external toll, such as the optimal congestion toll collected by a transportation authority, is required to attain the optimal allocation. Our privately owned and selfish profit maximizing transportation companies end up with charging the price exactly equivalent to the optimal toll. © 1998 Academic Press

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‡ Correspondence to: Faculty of Economics, Tokyo Metropolitan University, 1-1 Minami Osawa Hachioji, Tokyo 192-03, Japan. The authors' names are listed in alphabetical order.

## 1. INTRODUCTION

In this paper we consider a small open spatial economy with monocentric urban area where traffic congestion presents external diseconomy. We show that a competitive equilibrium allocation, without any governmental levy of congestion tolls, coincides with the social optimum allocation. The role of the well known optimal congestion tolls is replaced completely by profit maximizing competitive prices set by the transit service providers.

In order to prove our results, we first specify our model and derive the conditions for the competitive equilibrium allocation. In the competitive model, we assume that there is no government levying congestion tolls. Instead, we assume that there are many competing commuter transit services maximizing their own private profits, and we obtain a set of equilibrium conditions for the competitive economy. Then we turn to obtain socially optimum allocation. There, we assume a completely planned economy where the planner maximizes the social gains by choosing the controllable city variables such as allocation of land, population density, the amount of consumables, etc., subject to technical and market clearing conditions. By comparing the two set of conditions, we prove that the competitive allocation is compatible with the social optimal allocation.

Our result can be understood intuitively in the following manner. When there exist many competing transit service providers, passengers can select and use the service that offers the smallest gross travel cost, defined by the sum of congestion cost and pecuniary cost. As a result, competitive transit companies must take the gross travel cost as given, and choose their service prices and the amount of land used for transit service in order to maximize their profits. The price the competitive transit companies charge becomes identical to the amount of the optimal congestion toll. This competitive arrangement internalizes the congestion externality and leads to the social optimality.

Our work is an extension of Kanemoto ([12], Chap. 1) who proved the optimality of a competitive economy with monocentric urban model without congestion, a refinement of Hatta [6] who also proved a similar result in a model with traffic congestion. Our results should be distinguished from those of Oron, Pines, and Sheshinski [16], which required optimal congestion tolls levied by the transportation authority.

In addition, our results shed some light on the problem of efficient use of land for housing and transportation. For housing land, Oron, Pines, and Sheshinski [16] showed that the efficient land use for housing required the optimal congestion tolls and the competitive housing market. On the other hand, the problem of optimal land use for transportation was first studied by Mills and de Ferranti [15], and Solow and Vickrey [21]. It was further developed by Hochman [7], and Pines and Sadka [18]. These authors

essentially showed that the efficient land use for transportation required a cost benefit rule based on the market prices in addition to the optimal congestion tolls and the competitive housing market. In contrast, our results show that our competitive economy attains efficient land use for both housing and transportation without any congestion tolls nor cost benefit approach. Thus our competitive economy is able to replace the required optimal congestion toll and cost-benefit rule in order to obtain the efficient land use pattern.<sup>1</sup>

Section 2 lays out the basic framework of our model. The market equilibrium is defined in Section 2.2, and in Section 3 we derive the necessary conditions for social optimality. Section 4 proves the compatibility of the competitive and socially optimal allocations. A literature review is provided in Section 5. Some concluding remarks are given in Section 6.

## 2. THE COMPETITIVE EQUILIBRIUM

### 2.1. *The Model*

#### 2.1.1. *Consumers*

Consider a spatial economy, where urban and rural differences exist. The consumers, whether they live in or outside the city, have identical utility functions, and derive their utility by consuming two goods, consumables and housing space.

The utility function of a representative consumer is given by

$$u = u(z - c, h), \quad (2.1)$$

where  $z$ ,  $h$ , and  $c$  are the quantities of consumables, units of housing space consumed,<sup>2</sup> and the congestion cost that she suffers by commuting, respectively. Congestion is measured in units of the consumables, and it represents the disutility a passenger suffers from commuting. The utility function is increasing in both  $z - c$  and  $h$ .

Consumers are utility maximizers. They can move without any cost from one location to an other in order to achieve the highest possible utility.

*In the rural area.* Once a consumer decides to live in the rural area, she receives the rural wage rate, and has to pay for consumables and housing space she consumes. We set the price of consumables as the numeraire.

<sup>1</sup> See 'Note on the Literature' section (Section 5) for more detailed discussion of the literature.

<sup>2</sup> Since, in this model, no explicit construction of housing is introduced, the housing space,  $h$ , means the size of a plot of land.

The rural rental rate and the rural wage rate are  $r^a$  and  $w^a$ , respectively. They are assumed to be identical everywhere in the rural area. Therefore, the location where a consumer lives within the rural area does not effect her utility. There is no commuting in the rural area, so there is no congestion cost. These assumptions imply that the consumer maximizes

$$u = u(z^a, h^a) \quad (2.2)$$

subject to

$$w^a = z^a + r^a h^a, \quad (2.3)$$

where  $z^a$  and  $h^a$  are quantities of consumables and housing space consumed by the rural consumer, respectively. The maximization determines the quantities  $z^a$  and  $h^a$  as functions of  $w^a$  and  $r^a$ . Substituting these back in to (2.2) gives the indirect utility function of the form

$$u^a = v(w^a, r^a). \quad (2.4)$$

The small city assumption implies that the rural area is large in the sense that consumers can actually sell as much labor services as they want at the fixed wage rate  $w^a$ , and can purchase as large a housing space as possible at the fixed unit price  $r^a$ . Then the level of utility achieved by a rural resident is fixed at  $u^a$ .

*In the urban area.* On the other hand, if the consumer decides to live in the urban area, she has to commute to work in the central business district (CBD), and receives the urban wage rate  $w$ . She purchases consumables and pays for housing space as the rural residents do. In addition, she must pay for commuting to the CBD. The commuting cost depends on how far from the CBD she lives.

The distance from the CBD is measured by the variable  $x$ , and we say that she lives at  $x$ , meaning that she lives somewhere in the ring  $x$  miles away from the CBD. The commuting cost,  $t(x)$ , then is an increasing function of  $x$ . When she lives at  $x$ , her disposable income,  $w - t(x)$ , also becomes a function of  $x$ . Accordingly, the amount of consumables and housing space she can purchase depends on  $x$ , and we write them as  $z(x)$  and  $h(x)$ .

When she commutes from  $x$  to the CBD, the consumer suffers from total congestion given by the amount of  $c(x)$ . As the land supply at  $x$  in the city is limited, the housing rent must depend on  $x$  as well and it is written as  $r(x)$ .<sup>3</sup>

<sup>3</sup> We assume that there are absentee landlords who receive all the rent.

Putting these assumptions together, a city resident at  $x$  maximizes<sup>4</sup>

$$u = u(z(x) - c(x), h(x)) \quad (2.5)$$

subject to

$$w - t(x) = z(x) + r(x)h(x). \quad (2.6)$$

The utility maximization implies that

$$\frac{u_h(z(x) - c(x), h(x))}{u_z(z(x) - c(x), h(x))} = r(x). \quad (2.7)$$

Since the city is small<sup>5</sup> and consumers can move freely, the utility level of city residents everywhere in the city are identical and they must be identical to that of the rural residents,  $u^a$ , in equilibrium. This implies that

$$u(z(x) - c(x), h(x)) = u^a. \quad (2.8)$$

### 2.1.2. Urban Producers

Urban producers produce consumables at the CBD with a linear homogeneous production function given by  $y = F(N^*)$ , where  $N^*$  is the labor force employed. The profit maximization implies that

$$w = F'(N^*), \quad (2.9)$$

which is a constant due to the assumption of constant returns.

### 2.1.3. Commuter Transportation Industry

In order for the city residents to commute to the CBD, the transportation industry provides commuter train services.<sup>6</sup> The service provided at  $x$  is denoted by  $M(x)$ , meaning that  $M(x)$  is the number of the commuters who cross the ring at  $x$  from the CBD. Since  $c(x)$  is the total amount of congestion the commuter must face by commuting from  $x$  to the CBD, we can say that  $\dot{c}(x)$  is the instantaneous congestion at  $x$  and it is denoted by  $s(x)$ , that is

$$\dot{c}(x) = s(x). \quad (2.10)$$

<sup>4</sup> Notice that  $c(x)$  is external to the consumer.

<sup>5</sup> The smallness of the city is reflected in the fact that the rural rental and wage rates are given constants.

<sup>6</sup> Our model can be interpreted as a model of automobile roadway travel rather than a model of railroads. See Section 5 where our model is compared to the one by Oron, Pines, and Sheshinski [16].

We assume that at the CBD

$$c(\mathbf{0}) = 0. \quad (2.11)$$

The transportation industry produces rail services to transport  $M(x)$  of passengers at  $x$ . For that it requires an input of land,  $L(x)$ , of land when the congestion level at  $x$  is  $s(x)$ . Thus its production function is written as

$$M(x) = T(L(x), s(x)), \quad (2.12)$$

where  $T(\cdot)$  is assumed to be linear homogeneous in  $L(x)$ .<sup>7</sup>

In addition to the congestion cost, the commuter at  $x$  has to pay the pecuniary commuting cost  $t(x)$  to get to the CBD. We can define the instantaneous pecuniary cost  $p(x)$  such that

$$t(x) \equiv \int_0^x p(\tau) d\tau. \quad (2.13)$$

We call the sum of the congestion cost and pecuniary cost the instantaneous gross travel cost and denote it by  $g(x)$ , that is,

$$g(x) \equiv p(x) + s(x). \quad (2.14)$$

We assume that many privately owned rail service companies are competing with each other for commuter passengers. By competition, the gross travel cost,  $g(x)$ , not  $p(x)$ , must become identical to that of other companies. Each company maximizes its profit by selecting  $p(x)$  and  $L(x)$  regarding  $r(x)$  and  $g(x)$  as given.<sup>8</sup>

Using (2.12) and (2.14), the profit maximizing problem of each train company is written as

$$\text{Max}_{L(x), p(x)} \Pi \equiv \int_0^{\bar{x}} \{p(x)T(L(x), g(x) - p(x)) - r(x)L(x)\} dx.$$

Maximizing with respect to  $L(x)$  and  $p(x)$ , this yields

$$r(x) = p(x)T_L(L(x), s(x)), \quad (2.15)$$

and

$$M(x) = p(x)T_s(L(x), s(x)). \quad (2.16)$$

<sup>7</sup> The congestion level,  $s(x)$ , is external to the production process.

<sup>8</sup> Strictly speaking, we should differentiate companies by assigning the subscript  $i$  for the  $i$ th company. However, Hatta [6] showed that in the long run equilibrium the fare structure of each company becomes identical to each other when each firm has an identical production function. Thus we omitted the subscript from the beginning.

## 2.2. The Market Equilibrium Conditions

### 2.2.1. The City Labor Market

Let us denote the population density at  $x$  by  $N(x)$  and the city limit by  $\bar{x}$ . Since all the city residents work at the CBD, the total people employed,  $N^*$ , equals the total number of the city residents, that is

$$N^* \equiv \int_0^{\bar{x}} N(x) dx, \quad (2.17)$$

which is the equilibrium of the labor market.

### 2.2.2. The City Land Market

The space of land available in the city at  $x$  is denoted by  $\theta(x)$ .<sup>9</sup> The land demanded by the city residents and the transportation industry is  $h(x)N(x)$  and  $L(x)$ , respectively. Equilibrium in the land market requires

$$h(x)N(x) + L(x) = \theta(x). \quad (2.18)$$

Also the competitive bidding for land implies that at the city limit, we have

$$r(\bar{x}) = r^a. \quad (2.19)$$

### 2.2.3. The Commuter Train Service Market

The demand for transportation services at  $x$  is given by the total number of people who live in the band between  $x$  and  $\bar{x}$ . The equilibrium of the transportation market, therefore, is given by

$$M(x) = \int_x^{\bar{x}} N(\tau) d\tau,$$

which is identical to the following two equations.

$$\dot{M}(x) = -N(x), \quad (2.20)$$

$$M(x) = 0. \quad (2.21)$$

### 2.2.4. The Full Market Equilibrium Conditions

The conditions for the consumer's utility maximization are (2.6) and (2.7). The free mobility of consumers gives (2.8). Equation (2.9) is the urban producer's maximization condition. The profit maximization of the transportation industry implies conditions (2.12), (2.15), and (2.16). Also,

<sup>9</sup> If all topography of the urban area is inhabitable, that is, if there is no lakes, no oceans, or no mountains,  $\theta(x)$  is equal to  $2\pi x$ .

the definition of congestion and the rail service price provide us with (2.11), (2.10), and (2.13). The equilibrium of the labor market and the land market are attained by (2.17), (2.18), and (2.19). Also, the equilibrium of the transportation market is attained by (2.20) and (2.21).

In our model, there are ten unknown functions of  $x$ ,  $r(x)$ ,  $p(x)$ ,  $h(x)$ ,  $z(x)$ ,  $t(x)$ ,  $c(x)$ ,  $s(x)$ ,  $L(x)$ ,  $M(x)$ ,  $N(x)$ , and three unknown variables,  $w$ ,  $\bar{x}$ ,  $N^*$ , to be determined. On the other hand, there are ten equations which should hold true in each value of  $x$ , and five single equations. The fact that there are two extra single equations reflects the fact that (2.10) and (2.20) are differential equations, and that they require two initial conditions.

When these equations and variables are determined, we say that the market equilibrium is reached in the small open city.

### 3. THE SOCIAL OPTIMAL RESOURCE ALLOCATION

#### 3.1. *The Problem Stated*

The social optimal problem is solved by maximizing

$$J \equiv F(N^*) - \int_0^{\bar{x}} (z(x) - c(x))N(x) dx \\ - \int_0^{\bar{x}} c(x)N(x) dx - \int_0^{\bar{x}} r^a \theta(x) dx$$

subject to (2.8), (2.10), (2.12), (2.17), (2.18) and (2.20) with the boundary conditions (2.11) and (2.21).

This means maximizing the net product of the city, which is defined as the total product minus the net consumption of city residents, minus the total cost of congestion, minus the payment to the land owners evaluated by the rural rent. In other words, it is the net economic rent created by organizing the small open city. The process of maximization is constrained by the fact that we must guarantee that city residents achieve the same utility level as rural consumers at  $u^a$ , and that the supplies of labor, land, and transportation services in the city must be equal to their respective demands in order for the maximization to be feasible. We say that the social optimum is reached in the small open city when this total economic rent is maximized.<sup>10</sup>

$J$  can be simplified to

$$J = F(N^*) - \int_0^{\bar{x}} z(x)N(x) dx - \int_0^{\bar{x}} r^a \theta(x) dx.$$

<sup>10</sup> This is considered to be a case of the optimal control problem known as the problem of Hestenes and Bolza.

The first order conditions of this problem can be found from the following Lagrangian defined by

$$\begin{aligned} \mathcal{L} \equiv & F(N^*) + \int_0^{\bar{x}} [I(\cdot)] dx + \mu N^* \\ & - \lambda_1(\bar{x})M(\bar{x}) - \lambda_2(\bar{x})c(\bar{x}) + \lambda_1(\mathbf{0})M(\mathbf{0}) + \lambda_2(\mathbf{0})c(\mathbf{0}) \\ & + \gamma_2^0 c(\mathbf{0}) + \gamma_1^1 M(\bar{x}), \end{aligned}$$

where

$$\begin{aligned} I(\cdot) \equiv & I(z(x), h(x), L(x), c(x), N(x), M(x), s(x)) \\ \equiv & \dot{\lambda}_1(x)M(x) + \dot{\lambda}_2(x)c(x) - \lambda_1(x)N(x) + \lambda_2(x)s(x) \\ & + \{-z(x)N(x) - r^a\theta(x)\} + \nu(x)\{u(z(x) - c(x), h(x)) - u^a\} \\ & + \pi(x)\{T(L(x), s(x)) - M(x)\} - \mu N(x) \\ & + \gamma(x)\{\theta(x) - h(x)N(x) - L(x)\}. \end{aligned}$$

Notice that  $\nu(x)$ ,  $\lambda_2(x)$ ,  $\mu$ ,  $\gamma(x)$ ,  $\pi(x)$  and  $\lambda_1(x)$  are the Lagrangian multipliers associated with (2.8), (2.10), (2.12), (2.17), (2.18) and (2.20), respectively.

### 3.2. The Necessary First Order Conditions

Then the necessary conditions consist of five equations on the control variables, two adjoint equations on the state variables, and six transversality equations on the control parameters. They are summarized as follows

$$I_{s(x)} = \lambda_2(x) + \pi(x)T_s(L(x), s(x)) = 0, \quad (3.1)$$

$$I_{z(x)} = -N(x) + \nu(x)u_z(z(x) - c(x), h(x)) = 0, \quad (3.2)$$

$$I_{h(x)} = -\gamma(x)N(x) + \nu(x)u_h(z(x) - c(x), h(x)) = 0, \quad (3.3)$$

$$I_{L(x)} = -\gamma(x) + \pi(x)T_L(L(x), s(x)) = 0, \quad (3.4)$$

$$I_{N(x)} = -\lambda_1(x) - \mu + (-z(x) - \gamma(x)h(x)) = 0, \quad (3.5)$$

$$\mathcal{L}_{M(x)} = \dot{\lambda}_1(x) - \pi(x) = 0, \quad (3.6)$$

$$\mathcal{L}_{c(x)} = \dot{\lambda}_2(x) - \nu(x)u_z(z(x) - c(x), h(x)) = 0, \quad (3.7)$$

$$\mathcal{L}_{N^*} = F'(N^*) + \mu = 0, \quad (3.8)$$

$$\mathcal{L}_{M(\mathbf{0})} = \lambda_1(\mathbf{0}) = 0, \quad (3.9)$$

$$\mathcal{L}_{c(0)} = \lambda_2(\mathbf{0}) + \gamma_2^0 = \mathbf{0}, \quad (3.10)$$

$$\mathcal{L}_{M(\bar{x})} = -\lambda_1(\bar{x}) + \gamma_1^1 = \mathbf{0}, \quad (3.11)$$

$$\mathcal{L}_{c(\bar{x})} = -\lambda_2(\bar{x}) = \mathbf{0}, \quad (3.12)$$

$$\begin{aligned} \mathcal{L}_{\bar{x}} &= I(\bar{x}) - \dot{\lambda}_1(\bar{x})M(\bar{x}) - \dot{\lambda}_2(\bar{x})c(\bar{x}) \\ &\quad - \lambda_1(\bar{x})\dot{M}(\bar{x}) - \lambda_2(\bar{x})s(\bar{x}) + \gamma_1^1\dot{M}(\bar{x}) = \mathbf{0}. \end{aligned} \quad (3.13)$$

#### 4. THE COMPATIBILITY

In this section, we establish that the competitive equilibrium allocation coincides with the socially optimal allocation.

Let  $r(x) = \gamma(x)$  and  $p(x) = \pi(x)$ . In other words, we interpret  $\gamma(x)$  and  $\pi(x)$  to be the rental rate of the land and the price of the rail service charged at  $x$ , respectively.

##### 4.1. Utility Maximization of Urban Consumers

Equations (3.2) and (3.3) imply that

$$\frac{u_h(z(x) - c(x), h(x))}{u_z(z(x) - c(x), h(x))} = r(x). \quad (4.1)$$

Equations (3.5), (3.8) and (4.1) imply the consumer's utility maximization when the wage is paid by the marginal product of the CBD, and they correspond to (2.6), (2.7), and (2.9).

##### 4.2. Profit Maximization of Commuter Rail Services

Equations (3.2) and (3.7) imply that

$$\dot{\lambda}_2(x) = -N(x),$$

which, by integration, yields

$$-\int_x^{\bar{x}} N(\tau) d\tau = \int_x^{\bar{x}} \dot{\lambda}_2(\tau) d\tau.$$

Using the definition of  $M(x)$  together with (3.1) and (3.12), this becomes

$$M(x) = \pi(x)T_s(L(x), s(x)). \quad (4.2)$$

Equations (3.4) and (4.2) coincide with the profit maximization conditions of the transportation industry given by (2.15) and (2.16).

### 4.3. The Optimum City Size

A straight forward calculation of (3.13) with (3.11) and (3.12) yields

$$\begin{aligned} \mathcal{L}_{\bar{x}} &= \{-z(\bar{x})N(\bar{x}) - r^a\theta(\bar{x})\} + \nu(\bar{x})\{u(z(\bar{x}) - c(\bar{x}), h(\bar{x})) - \bar{u}\} \\ &\quad + \pi(\bar{x})\{T(L(\bar{x}), s(\bar{x})) - M(\bar{x})\} \\ &\quad + \gamma(\bar{x})\{\theta(\bar{x}) - h(\bar{x})N(\bar{x}) - L(\bar{x})\} - \{\lambda_1(\bar{x}) + \mu\}N(\bar{x}) \\ &= \{-\lambda_1(\bar{x}) - \mu - z(\bar{x}) - \gamma(\bar{x})h(\bar{x})\}N(\bar{x}) + \{-r^a + \gamma(\bar{x})\}\theta(\bar{x}) \\ &\quad + \{-\gamma(\bar{x})L(\bar{x}) + \pi(\bar{x})T(L(\bar{x}), s(\bar{x}))\}. \end{aligned}$$

The first term of the last expression is zero from (3.5). Also, the third term of the expression is zero due to (3.4) and homogeneity of  $T(\cdot)$  with respect to  $L(x)$ . Thus, we obtain

$$r^a = \gamma(\bar{x}). \quad (4.3)$$

Equation (4.3) corresponds to (2.19), which determines the border of the city,  $\bar{x}$ .

### 4.4. Other Conditions

From (3.6) and (3.9), we obtain

$$\int_0^x \pi(\tau) d\tau = \lambda_1(x). \quad (4.4)$$

When  $t(x)$  is regarded as  $\lambda_1(x)$ , Equation (4.4) is identical to (2.13).

Equations (2.8), (2.10), (2.12), (2.17), (2.18), and (2.20) are used as constraints of the social optimal problem and hence, though they are not listed in Section 3.2, they are a part of the first order conditions of the optimal problem.

Equations (2.11) and (2.21) are the boundary conditions, and they belong to the optimal system as well.

In short, all of the equilibrium conditions in Section 2.2 correspond to the necessary conditions of the social optimum problem.

We are now able to state the following theorem.

**THEOREM 1.** *The competitive market equilibrium conditions are identical to the necessary first order conditions of the social optimal problem.*<sup>11</sup>

In relation to the optimal land use problem for housing and transportation, we can observe the following. In our model, if the amount of land

<sup>11</sup> We can interpret our result as an application of the more general principle shown in club good models. See Buchanan [4] and Berglas [2] for more details on this literature. We owe this observation to one of the referees.

used for transportation,  $L(x)$ , is an exogenous function, the market equilibrium conditions conditional on exogenous  $L(x)$  continue to be identical to the ones of social optimal problem conditional on exogenous  $L(x)$ . This can be seen easily. By setting  $L(x)$  exogenous, we must throw away equation (2.15) from our competitive market equilibrium conditions. From the social optimal problem, the optimality condition on  $L(x)$ , i.e. equation (3.4), must also be taken out. The remaining conditions continue to hold and thus we establish the following theorem.

**THEOREM 2.** *If the amount of land allocated for transportation,  $L(x)$ , is an exogenous function, the remaining market equilibrium conditions are identical to the social optimal conditions conditional on exogenously given  $L(x)$ .*

These theorems imply two results with regard to the efficient land use problem.<sup>12</sup> First result is due to Theorem 2. We can see that our profit maximizing road or railroad owners, if competitive, will internalize the congestion externality by imposing a fare, as a function of distance from the CBD, equivalent to the optimal congestion toll.<sup>13</sup> And therefore, the residential land use is efficient whether or not the land use for transportation is efficient. Second result is due to Theorem 1. That is, if  $L(x)$  is endogenous, profit maximizing road or railroad owners, if competitive, will use the efficient amount of land for transportation. Thus we can attain efficient land use for transportation without any external congestion tolls imposed by the government authority or any cost-benefit rule.

The intuition of this should be clear. The profit maximizing condition with respect to  $L(x)$ , Equation (2.15), is identical to the cost benefit rule based on the market prices. The left hand side is urban housing rent and it is the marginal social cost of converting a unit of housing land into transportation land. The right hand side is the marginal social benefit of extra number of people passing through  $x$  by the additional transportation land. The equation shows that the land for transportation is allocated to the point that the marginal social cost is equal to the marginal social benefit. These results complement the existing efficiency results in the literature.<sup>14</sup>

## 5. NOTE ON THE LITERATURE

The first study on the relationship between the congestion externality and the optimal toll appeared in Pigou [17]. He presented an example, known as “the case of two roads,” to demonstrate that a public investment

<sup>12</sup> We are indebted to one of the referees to point this out.

<sup>13</sup> See Section 5 for equivalence of the competitive fare and the optimal congestion toll.

<sup>14</sup> See Section 5 for more detailed discussion of the literature.

to construct a new road could end up in futility if the external economy were to create traffic congestion on the road. He called for a tax which covers the marginal social cost of externality to secure the maximum efficiency in the use of the two roads.

Knight [13] contested the Pigou's contention by proving that if the ownership of the road is given to a private sector agent, precisely the ideal situation, which was identical to the one brought out by the Pigouvian tax, would be established through the operation of ordinary economic motives. Faced with this claim, Pigou withdrew his example from his last edition of the *Economics of Welfare*.

Strotz [22] first studied the effect of transport congestion in the context of an optimal urban land use model. Oron, Pines, and Sheshinski [16] studied the optimum and equilibrium land use pattern for housing in a continuous monocentric city model with a traffic congestion where the transportation land was predetermined, and proved the following proposition.<sup>15</sup>

**PROPOSITION 1.** *In a monocentric urban economy model where the transportation land is predetermined, the optimum resource allocation is achieved and can be supported by a competitive price system if a government authority imposes optimal congestion tolls, which is equal to the number of households commuting through  $x$  to the center of the city,  $M(x)$ , times the time spent due to the marginal increase in commuter,  $t_M$ , and redistributes the proceed as a lump sum subsidy.*

Despite their apparent differences, our model is quite similar to the OPS model. For example we prefer to put  $c(x)$  (and  $s(x)$ ), the congestion variable, explicitly in utility and production functions. In doing so, we interpret the congestion in our model to be the physical fatigue by riding commuter trains. The speed of the train is invariant to whether the congestion existed or not. The congestion in our model, therefore can be understood as the extra calories needed by consumers to get to work when riding in the crowded train. And higher level of congestion does not crowd out the workers from reaching the city center. For the train service operators, the congestion is a factor that represents the quality of their train service. If the level of congestion is high, passengers will move away from the crowded line to a less congested line.

In the OPS model, on the other hand, the congestion is formulated as the time cost for consumers and the reduced speed in transportation services. The congestion so formulated is a time loss for consumers and it could crowd out the commuters from reaching the city center within a

<sup>15</sup> Pines and Sadka [18] reaffirmed the result by using the model developed by Arnott [1], where the width of the road was parametrically given.

limited time. However, the formality of our model becomes almost identical to the OPS model if some substitutions and reinterpretations are made. For instance, if the OPS's  $Z(x) + \tau(x)/P_z$  is replaced by our  $z(x)$  and their  $\tau(x)/P_z$  is replaced by our  $c(x)$ , and if they are reinterpreted accordingly, the formal model structure of our consumer sector becomes identical to that of the OPS model.

A similar transformation will prove that the transportation sector of the OPS model is formally identical to ours if their velocity function,  $V(x) = v(E(x), G_2(x))$ , is rewritten as

$$E(x) = f(1/V(x), G_2(x)), \quad (5.1)$$

where  $E(x)$  is the number of commuters who pass through  $x$ ,  $V(x)$  is the velocity of moving those passengers, and  $G_2(x)$  is the land allocated for the use of transportation. Notice that their  $E(x)$  is identical to our  $M(x)$  and their  $G_2(x)$  is also identical to our  $L(x)$ . Therefore equation (5.1) is formally identical to our transportation production function if  $1/V(x)$  is interpreted as our  $s(x)$ .

There is one difference in treating the land for transportation. As we mentioned earlier, OPS assumes that  $G_2(x)$  is a predetermined function of  $x$ . However, in our model,  $L(x)$  is one of the endogenously determined functions (Theorem 1) and it can be considered exogenous if necessary (Theorem 2).

But it is the formulation of the behavioral assumptions that makes two models distinct from each other. The difference can be compared to that of Pigou and Knight. The OPS theorem states that the optimal toll must be collected by the transportation authority (the TA hereafter) to attain efficiency and that the equilibrium is supported by the competitive price system under the condition that the TA behaves as a tax authority collecting the optimal external cost. The TA is not a private selfish enterprise that seeks maximum profit. This is Pigou at heart.

On the other hand, we formulate the competitive behavior of the transportation sector as a privately owned and selfish profit maximizer. This makes sense only if there are quite a few train service providers competing for passengers in order to maximize their profits. The resulting profit maximizing price  $p(x)$  is given by our Equation (2.16), which can be rewritten to give

$$p(x) = M(x)/T_s. \quad (5.2)$$

Since  $1/T_s$  can be interpreted as  $ds/dT$ ,  $p(x)$  can be interpreted as the marginal external congestion cost of an additional commuter. Therefore, the optimal toll of the OPS becomes identical to our competitive price.

This should be so since we proved that the economy attains the optimal resource allocation under this price.

In view from the efficient land use problem, the OPS theorem proved that residential land use was efficient if the optimal congestion tolls were collected and redistributed as a lump sum subsidy under the competitive residential land market. Our Theorem 2, on the other hand, states that profit maximizing road/railroad owners, if competitive, will impose a fare equivalent to the optimal congestion toll, and thus residential land use is efficient whether or not the division of land between residential use and transportation use is efficient. Clearly our approach belongs to that of Knight.<sup>16</sup>

Mills and de Ferranti [15] were the first to examine the optimal land use pattern for roads in the context of a continuous monocentric city with fixed housing lot size. They obtained a mathematical expression for the optimality.

Hochman [7] extended Mills and de Ferranti result by introducing congestion tolls paid by the commuters. Maintaining the assumption of a fixed housing lot size, Hochman proved that the Mills and de Ferranti efficiency condition could be attained if each commuter paid the "full congestion toll," which was equal to per capita market rent of the land used for transit service. His "full congestion toll" can be easily shown to be equivalent to the social marginal cost of congestion. Therefore, the commuters in the Hochman's model must pay the congestion toll equivalent to the optimal tolls derived by the OPS model. The toll must be collected by government authority in practice.

In addition to the optimal toll, the Hochman's result required the condition that equated the market housing rent to the marginal increase of transit service due to a marginal addition of land. The condition may be interpreted as the cost-benefit criterion by which to allocate the land to the transit service. In this way the following proposition was first indicated by Hochman. It was shown more clearly later, but not stated in words, by Pines and Sadka [18].

*PROPOSITION 2. If a government authority imposes optimal congestion tolls, then a simple cost-benefit rule based on comparing the market rent of residential land to the direct benefit of added transportation service from*

<sup>16</sup> Hochman and Pines [8] stated in their concluding remarks section that their model (which included traffic congestion and privately owned profit maximizing transit service sector) produced an efficient outcome. However, they did not provide the proof. If the HP model produces the efficiency result, it certainly is a model that belongs to the approach of Knight. For that, they need to prove the efficiency and should show that their consumers end up paying the fare equivalent to the social external cost of congestion.

*adding land to transportation use leads the government to devote the efficient amount of land to transportation.*

Our Theorem 1 proved that the roles of the optimal congestion tolls and the cost-benefit criterion in Proposition 2 can be dropped and replaced by the competitive behavior of our transit service providers. And as a result the efficient use of land for transportation is assured.<sup>17</sup>

As we mentioned earlier, we can understand our result intuitively. Our profit maximizing transit providers use land efficiently as shown by equation (2.15), which is essentially equivalent to the cost-benefit condition since the left hand side is the market rent of housing land and the right hand side is the marginal benefit of added transportation service from an additional land to the service. In addition, equation (2.16) holds, which in turn gives equation (5.2). Thus the consumers pay the train fare that is equivalent to the marginal external cost of congestion.

Hatta [6] proved a similar result to ours in a model with traffic congestion. In his model, the consumers were assumed to have identical utility functions which depend only on one good called consumables. In other words, his model assumed away the possibility of substitution between consumables and housing land, and forced the consumers to use a fixed amount of housing land. In order to compensate for this, he introduced capital in his model, which made his model unnecessarily complex. Therefore his result was not readily comparable to the orthodox competitive models. Our model treats housing land as one of the consumer's choice variables and produces a result readily comparable to the standard optimality result of the competitive economy.

## 6. CONCLUDING REMARKS

We have shown that competitive forces will bring about the social optimal allocations in a small open city model where traffic congestion arises as an external economy to consumers and to the transportation industry as well. The present results depend on the assumptions that the production function of consumables is linear homogeneous and that the providers of the commuter services compete with each other for passengers on the basis of the congestion inclusive price of transportation services.

<sup>17</sup> Notice that the present paper is directly related to the first best resource allocation problem where the congestion is optimally priced (or optimally priceable) by government authority. There are quite a few papers in the literature that examine the resource allocation problem in the second best (or third best) environment. They include Solow [20], Kanemoto [10, 11], Robson [19], Arnott [1], and Pines and Sadka [18]. See also Braid [3] for a grand listing of papers that consider the optimal land use problem.

Though we have proved the efficiency results in an open small city model, we can readily extend the result to the case of a full general equilibrium model where the rural variables are determined jointly with the urban variables.<sup>18</sup>

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<sup>18</sup> In fact Fukushima and Shigeno [5] just do this by assuming that all consumers have equal share to land ownership, resulting in equal redistribution of rental income for every consumer.

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